# VARIATONAL ITERATION METHOD FOR SOLUTION OF ONE-DIMENSIONAL HEAT EQUATION WITH NONLOCAL CONDITIONS

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**Abstract**

The present paper discusses the closed-form solution of the one dimensional Heat equation with Nonlocal condition by Variational Iteration Method (VIM). The advantage of this method is to overcome the difficulty of calculation of Adomian's polynomials in the Adomian's decomposition method. Variational Iteration Method has been widely used to handle linear and nonlinear models. Numerical results are provided to show the accuracy of the proposed method.

**Keywords**: Variational Iteration Method, Heat Equation, Nonlocal conditions.

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# 1 Introduction

The present paper is concerned with the closed form solution of heat equation with nonlocal boundary conditions. Over the last few years many physical phenomena were formulated into nonlocal mathematical models. Hence the solution of parabolic partial differential equations with nonlocal boundary conditions is currently an interesting area of research. The nonlocal problems are very important in the transport of reactive and passive contaminates in aquifers. These types of problems arise in quasi-static theory of thermo elasticity [3]. Many Researchers such as Borhanifar [3], W. T. Ang [1] and Bahuguna [2] have solved the problem given in this paper with different methods such as Adomain decomposition method, Laplace transform and Finite difference method respectively. We have solved this problem by variational iteration method which is very easy to implement and gives accurate solution of the problem.

The variation iteration method was developed in 1999 by He. This method is, now, widely used by many researchers to study linear and nonlinear problems. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications. It is based on Lagrange multiplier and it has the merits of simplicity and easy execution. It was shown by many authors ([6], [7], [8]) that this method is more powerful than existing techniques such as the Adomain decomposition method, perturbation method, etc. The VIM gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purpose.

The purpose of the paper is to use the variational iteration method for solving following one dimensional heat equation with non local boundary conditions this type of problem arising in the quasi-static theory of thermo-elasticity[2,4,5,9].

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=g(x,t) \quad \text{for } x\in(0,1) \quad \text{and } t>0 \tag{1}$$

Subject to the initial condition

$$u(x,0)=\psi(x) \quad \text{for } x\in(0,1) \tag{2}$$

and the non-local boundary conditions

$$\left.\begin{aligned} u(0,t) &= \int_0^1 p(x)\,u(x,t)\,dx + r(t) \\ u(1,t) &= \int_0^1 q(x)\,u(x,t)\,dx + v(t) \end{aligned}\right\} \text{ for } t>0, \tag{3}$$

where $x$ and $t$ are the spatial and time coordinates respectively, $u$ is the unknown function of $x$ and $t$ to be determined and $g, \psi, p, q, r$ and $v$ are suitably prescribed functions.

## 2 Variational Iteration Method

Consider the following differential equation:

$$Lu + Nu = g(x,t) \tag{4}$$

where $L$ is a linear operator, $N$ a nonlinear operator, and $h(x,t)$ is the source inhomogeneous term. The VIM was proposed by "J.H. He (1999)", where a correctional functional for equation (7) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))\,d\tau\,, n \geq 0 \tag{5}$$

Where $\lambda$ is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript $n$ indicates the $n^{th}$ approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. ([6],[8]). It is clear that the successive approximations $u_n, n \geq 0$ can be established by determining $\lambda$, so we first determine the Lagrange multiplier $\lambda$ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained using the Lagrange multiplier obtained and by using any selective function $u_0$. The initial values $u(x,0)$ and $u_t(x,0)$ are usually used for selecting the zeroth approximation $u_0$ .With $\lambda$ determined, then several approximation $u_n(x,t),\ n\geq 0$, follow immediately. Consequently the exact solution may be obtained by using

$$u=\lim_{n\to\infty} u_n \tag{6}$$

# 3 Numerical Examples

In this section, we apply the Variational Iterational Method in the following examples.

**Example 1:** In this example [2], we consider (1) with

$$\begin{aligned} &g(x,t)=x(x-1)e^{-t},\psi(x)=x(x-1)-2,\\ &p(x)=\frac{12}{13},q(x)=\frac{12}{13},\ r(t)=0,v(t)=0 \end{aligned} \tag{7}$$

Now following the variational iteration method given in the above section, stationary conditions can be obtained as follows:

$$\begin{aligned} &\lambda'(\tau)=0\\ &1+\lambda(\tau)\big|_{\tau=t} \end{aligned} \tag{8}$$

The Lagrange multiplier can therefore be simply identified as $\lambda=-1$, and substituting this value of Lagrange multiplier into the functional (5) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\tau)}{\partial t}-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}-g(x,\tau)\right)d\tau \tag{9}$$

As stated before, we can select Initial condition given in the equation (7) and using this selection in (9) we obtain the following successive approximations:

$$u(x,0)=u_0(x)=\psi(x)=x(x-1)-2$$

$$u_1(x,t) = -2+2t+(-1+x)x+(-1+x)x(-1+Cosh[t]-Sinh[t]) \tag{10}$$

$$\begin{aligned} u_2(x,t) &= -x+(-1+x)x+x^2-2Cosh[t]-(-1+x)xCosh[t] \\ &+2(-1+x)x(-1+Cosh[t]-Sinh[t])+2Sinh[t]+(-1+x)xSinh[t] \end{aligned} \tag{11}$$

$$\begin{aligned} u_3(x,t) &= 2-x+(-1+x)x+x^2-(-1+e^{-t})(-2+x)(1+x) \\ &-4Cosh[t]-(-1+x)xCosh[t]+3(-1+x)x(-1+Cosh[t]-Sinh[t]) \\ &+4Sinh[t]+(-1+x)xSinh[t] \end{aligned} \tag{12}$$

$$\begin{aligned} u_4(x,t) &= 2-x+(-1+x)x+x^2-2(-1+e^{-t})(-2+x)(1+x) \\ &-4Cosh[t]-(-1+x)xCosh[t]+4(-1+x)x(-1+Cosh[t]-Sinh[t]) \\ &+4Sinh[t]+(-1+x)xSinh[t]+2(1-Cosh[t]+Sinh[t]) \end{aligned} \tag{13}$$

$$\begin{aligned} u_5(x,t) &= 2-x+(-1+x)x+x^2-3(-1+e^{-t})(-2+x)(1+x) \\ &-4Cosh[t]-(-1+x)xCosh[t]+5(-1+x)x(-1+Cosh[t]-Sinh[t]) \\ &+4Sinh[t]+(-1+x)xSinh[t]+4(1-Cosh[t]+Sinh[t]) \end{aligned} \tag{14}$$

$$\begin{aligned} u_6(x,t) &= 2-x+(-1+x)x+x^2-4(-1+e^{-t})(-2+x)(1+x)- \\ &4Cosh[t]-(-1+x)xCosh[t]+6(-1+x)x(-1+Cosh[t]-Sinh[t])+ \\ &4Sinh[t]+(-1+x)xSinh[t]+6(1-Cosh[t]+Sinh[t]) \end{aligned} \tag{15}$$

Simplifying equation (15) we get,

$$u_6(x,t)=e^{-t}(-2+x)(1+x) \tag{16}$$

$$u_7(x,t)=e^{-t}(-2+x)(1+x) \tag{17}$$

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$$u_n(x,t)=e^{-t}(-2+x)(1+x) \tag{18}$$

Hence the closed form solution of (1) with (7) is given as

$$u(x,t)=e^{-t}(-2+x)(1+x) \tag{19}$$

**Example 2:** In this example [2], we consider (1) with

$$\begin{aligned} g(x,t) &= \frac{-2(x^2+t+1)}{(t+1)^3} \\ \psi(x) &= x^2,\ p(x)=x,\ q(x)=x \\ r(t) &= \frac{-1}{4(t+1)^2},\ v(t)=\frac{3}{(t+1)^2} \end{aligned} \tag{20}$$

Now following the variational iteration method given in the above section, stationary conditions can be obtained as follows:

$$\begin{aligned} &\lambda'(\tau)=0 \\ &1+\lambda(\tau)\big|_{\tau=t} \end{aligned} \tag{21}$$

The Lagrange multiplier can therefore be simply identified as $\lambda=-1$, and substituting this value of Lagrange multiplier into the functional (5) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n(x,\tau)}{\partial t}-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}-g(x,\tau)\right)d\tau \tag{22}$$

As stated before, we can select Initial condition given in the equation (7) and using this selection in (22) we obtain the following successive approximations:

$$u(x,0)=u_0(x)=\psi(x)=x^2$$

$$u_1(x,t)=2t+x^2-2\left(1+\frac{x^2}{2}-\frac{2+2t+x^2}{2(1+t)^2}\right) \tag{23}$$

$$u_2(x,t) = 2t + \frac{2t}{1+t} + x^2 - \frac{t(-2x^2 + t(2+2t-x^2))}{(1+t)^2} - 4(1 + \frac{x^2}{2} - \frac{2+2t+x^2}{2(1+t)^2}) \tag{24}$$

$$u_3(x,t) = 2t + \frac{4t}{1+t} + x^2 + 2(\frac{1}{2} - \frac{1}{2(1+t)^2})x^2 - \frac{t(-2x^2 + t(2+2t-x^2))}{(1+t)^2} - 6(1 + \frac{x^2}{2} - \frac{2+2t+x^2}{2(1+t)^2}) \tag{25}$$

$$u_4(x,t) = 2t + \frac{6t}{1+t} + x^2 + 4(\frac{1}{2} - \frac{1}{2(1+t)^2})x^2 - \frac{t(-2x^2 + t(2+2t-x^2))}{(1+t)^2} - 8(1 + \frac{x^2}{2} - \frac{2+2t+x^2}{2(1+t)^2}) \tag{26}$$

Simplifying equation (27) we get,

$$u_4(x,t) = \frac{x^2}{(1+t)^2} \tag{27}$$

$$u_5(x,t) = \frac{x^2}{(1+t)^2} \tag{28}$$

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$$u_n(x,t) = \frac{x^2}{(1+t)^2} \tag{29}$$

Hence the closed form solution of (1) with (20) is given as

$$u\ \ x,t\ \ = \left(\frac{x}{1+t}\right)^2 \tag{30}$$

**Conclusion:**

The above two study problem shows that the computation of the solution is very easy. The present method is very effective and takes very less time in comparison with other analytical method such as Adomain decomposition method. Variational iteration method overcomes the difficulty of calculation of the adomain polynomials in adomain decomposition method.

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